

## On representations of super coalgebras

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1994 J. Phys. A: Math. Gen. 27 6421

(<http://iopscience.iop.org/0305-4470/27/19/015>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

### Download details:

IP Address: 171.66.16.68

The article was downloaded on 01/06/2010 at 23:16

Please note that [terms and conditions apply](#).

# On representations of super coalgebras

Andreas Hüffmann

Department of Mathematics, King's College, Strand, London WC2R 2LS, UK

Received 18 March 1994

**Abstract.** The general structure of the representation theory of a  $\mathbb{Z}_2$ -graded coalgebra is discussed. The result contains the structure of Fourier analysis on compact supergroups and quantizations thereof as a special case. The general linear supergroups serve as an explicit illustration and the simplest example is carried out in detail.

## 1. Introduction

Although the theory of Fourier analysis on supergroups or related coset supermanifolds has received considerable attention, the results are still far from a systematic theory that extends the classical theory for Lie groups and symmetric spaces. Recent progress concerning spherical Fourier analysis on certain non-compact Riemannian super coset spaces has been made by Zimbauer [1, 2]. His work was directly motivated by the demands of supersymmetric field theoretical models for an electron moving in a disordered environment [3, 4]. The peculiarities of these theories stimulated the desire for a better understanding of compact supergroups as well. These appear, for instance, as angle variables upon the introduction of polar coordinates in supersymmetric models.

A systematic study of compact unitary supergroups finally led to the general setting that is to be presented in this paper. It is basically a structure theorem for graded coalgebras with impact not only on supergroups but also on super quantum groups or graded Hopf algebras, respectively. The arguments are simple and were finally settled in the foundations on the level of Zorn's lemma. After this work had been completed, Green's treatment of locally finite representations came to my attention [5]. It contains the structure theorem, derived by different arguments without exploiting the possibility of a grading.

The text falls into three major units. First, the general structure of graded coalgebras and comodules is discussed. This reveals the basic structure of Fourier analysis of graded coalgebras. The role of induced representations is outlined. Second, the coalgebra corresponding to locally finite modules of the general linear supergroup is analysed; Kac modules, Kac filtrations and their role for the structure of its representations are discussed. Finally the general theory is illustrated by the simplest examples. Here a coalgebra related to the Lie superalgebra  $gl(1, 1; \mathbb{C})$  is considered in detail. In spite of its very simple nature almost all general phenomena can be observed with a minimal computational effort. Contact is made with existing literature on  $gl(2, 1; \mathbb{C})$  and a summary is given.

Although the results of section 2 extend further, the convention that the term 'graded' will always mean ' $\mathbb{Z}_2$ -graded' is adopted as a framework. Since all objects come in a graded context the explicit annotation of 'graded' will be dropped unless in cases where some confusion may arise. For example '(semi)simple' has the meaning of 'graded (semi)simple'.

If a simple comodule is required to be irreducible in the non-graded sense this will be emphasized by using 'absolutely simple'. Furthermore, a basis is to be considered as a homogeneous basis of some comodule and homomorphisms are homogeneous of degree zero. The terms 'comodule' and 'representation' will be used synonymously.

Since the intention was not to extensively reproduce previously published material on Lie superalgebras, the text is kept informal at some stages. For instance root systems and weight bases are not discussed in detail and reference to the literature is given whenever these notions are needed to put a subject into a wider perspective. However, the basic line of thought is presented in a self-contained manner and should be readable without in depth knowledge of the vast literature on supergroups and superalgebras.

## 2. Representations of graded coalgebras

I first recall the most basic consequences from the definition of a coalgebra  $C$ . These follow directly from its two structure homomorphisms, that is a coassociative comultiplication  $\Delta : C \rightarrow C \otimes C$  and a counit  $\varepsilon : C \rightarrow k$ , which is a mapping into the base field  $k$ . To be explicit, coassociativity means  $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$  and the counit satisfies  $(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id$ ,  $\cdot$  being the multiplication with scalars.  $C$  is considered as a (right- or left-) comodule over itself, with structure map  $\Delta : C \rightarrow C \otimes C$ .

The coassociativity is responsible for the coalgebra being locally finite-dimensional [6]. That means, if  $x \in C$  is arbitrary but fixed, then there is a finite-dimensional subcomodule  $V$  such that  $x \in V \subset C$ . The simple proof is as follows. Let  $\{b_j\}_{j \in J} \subset C$  be a basis,  $\Delta(b_j) = \sum_{k \in J} c_{jk} \otimes b_k$ . Fix  $x \in C$ . Write  $\Delta(x) = \sum_{j \in J_x} c_j \otimes b_j$  with  $J_x \subset J$  finite. Coassociativity means  $\sum_{j \in J_x} \Delta(c_j) \otimes b_j = (\Delta \otimes id)\Delta(x) = (id \otimes \Delta)\Delta(x) = \sum_{k \in J} (\sum_{j \in J_x} c_j \otimes c_{jk}) \otimes b_k$ . Hence the span of  $\{c_j\}_{j \in J_x} \cup \{x\}$  is a finite-dimensional subcomodule of  $C$ .

The existence of a counit implies that every subcomodule of  $C$  is contained in the span of its coefficients; in other words, a coalgebra is the direct limit of finite coalgebras formed by the coefficients of its finite-dimensional comodules. For that let  $\{v_i\}_{i \in I}$  be a basis of a subcomodule  $V \subset C$  and  $\Delta(v_i) = \sum_{j \in I} v_j \otimes a_j^i$ . Then  $v_i = (\varepsilon \otimes id)\Delta(v_i) = \sum_{j \in I} \varepsilon(v_j) a_j^i$ .

The definition of a comodules structure map, say  $\beta : V \rightarrow V \otimes C$ , forces  $V$  to be included in a direct sum of copies of  $C$ . Injectivity is immediate from the left inverse of  $\beta$ ,  $(id \otimes \varepsilon)\beta = id$ . The fact that  $\beta$  is a comodule map is simply  $(\beta \otimes id)\beta = (id \otimes \Delta)\beta$  where  $id \otimes \Delta$  is defined as the comodule structure on  $V \otimes C$ . It should be noted that this is not a tensor product of representations, which is not defined unless the coalgebra is endowed with a compatible algebra structure. For a finite-dimensional  $V$ ,  $\dim(V) = n$ , it means an inclusion  $\beta : V \hookrightarrow C^n$  [6]. Moreover, all comodules are locally finite and the problem of finding all finite-dimensional representations has as its first step the question of how to decompose  $C$ .

To make contact with the physical terminology one should imagine  $C$  as some sort of functions on a quantum semigroup. Passing to a quantum group in general requires enlarging  $C$ . This has to be done since the existence of an antipode demands the invertibility of matrices, which might be impossible within  $C$  itself. The natural approach seems to be Manin's construction of Hopf envelopes [7]. The fundamental problem is the Fourier analysis of  $C$ , which may drastically change in its nature with the passage to a Hopf envelope. This will appear explicitly in the discussion of the general linear Lie superalgebra.

From the point of view of category theory, attempts to study representations lead to the investigation of injective objects. The structure theory of coalgebras appears to be a direct

generalization of the representation theory of finite groups. It is beautifully developed in Green's work on locally finite representations [5]. However, this aspect seems not to have been considered within the mathematical physics literature. For that a brief discussion will be given. It is based on Zorn's lemma as an alternative to the approach in [5].

For a  $C$ -comodule  $W$  let  $\sigma(W)$  be its socle, that is its semisimple subcomodule. For every semisimple  $V$  let  $F_V$  be the family of comodules the socles of which are  $V$ .  $F_V$  has a partial ordering by inclusions. The following lemma gives the basic intuition about the representation theory of  $C$ . For its proof note that a homomorphism of comodules is injective if and only if its restriction to the socle is injective.

*Lemma 1.* Let  $V$  be a semisimple  $C$ -comodule. The following assertions are equivalent and hold for every coalgebra.

- (i)  $F_V$  has a greatest element.
- (ii) Every totally ordered subset of  $F_V$  has an upper bound in  $F_V$ .
- (iii) There is a maximal element in  $F_V$ .
- (iv) There is a  $X \in F_V$  such that  $X \subseteq W$  always implies  $W \simeq X \oplus X'$ .
- (v) For every semisimple  $X$  let  $I_X \in F_X$  be a fixed choice of an element satisfying (iv). Then for every comodule  $W$  one has  $W \hookrightarrow I_{\sigma(W)}$ .

*Proof.* (i)  $\Rightarrow$  (ii) is trivial and (ii)  $\Rightarrow$  (iii) is Zorn's lemma.

(iii)  $\Rightarrow$  (iv) Let  $X \in F_V$  be a maximal element,  $X \subseteq W$ . Let  $X' \subset W$  be a maximal complementary subcomodule to  $X$  in  $W$ . Then  $X'$  is also maximal complementary to  $V \subseteq X \subseteq W$ . Hence  $W/X' \in F_V$ . But  $X \subseteq W/X'$  via  $x \mapsto x + X'$  and by maximality  $X = W/X'$ . Thus  $0 \rightarrow X' \rightarrow W \rightarrow W/X' \rightarrow 0$  splits and  $W \simeq X \oplus X'$ .

(iv)  $\Rightarrow$  (v) For a comodule  $W$  let  $\delta : \sigma(W) \rightarrow W \oplus I_{\sigma(W)}$  be given by  $x \mapsto \delta(x) = (x, x)$ . Observe that  $W \subseteq (W \oplus I_{\sigma(W)})/\delta(\sigma(W))$  and by (iv),  $I_{\sigma(W)} \subseteq (W \oplus I_{\sigma(W)})/\delta(\sigma(W)) \simeq W/\sigma(W) \oplus I_{\sigma(W)}$ . Since  $x \in \sigma(W)$  maps to  $(x, 0) + \delta(\sigma(W)) = (0, -x) + \delta(\sigma(W))$ , which is in the image of  $I_{\sigma(W)}$ , it is possible to pass to the quotient with respect to  $W/\sigma(W)$  and  $W \hookrightarrow I_{\sigma(W)}$ .

(v)  $\Rightarrow$  (i) The last observation holds especially for all maximal elements in  $F_{\sigma(W)}$ ;  $I_{\sigma(W)}$  is a greatest element.

Property (ii) is immediately verified by constructing an upper bound as a quotient of the direct sum of all comodules of a totally ordered subset, i.e. by forming its direct limit. □

The proof of lemma 1 is basically category theoretical and relies on the facts that comodules always have a socle and that factor comodules exist. In the language of category theory a comodule with the property (iv), i.e. a comodule that splits from every comodule that contains it as a subcomodule, is called an injective. Clearly any direct summand of an injective is also an injective. Using this terminology and given a full set of simple  $C$ -comodules  $F_0$  choose a corresponding set of injective covers  $F$ , that is a set of undecomposable injective comodules the socles of which are in  $F_0$ .

For every comodule  $\beta : V \rightarrow V \otimes C$  there is a coefficient mapping

$$\phi_V : V^* \otimes V \rightarrow C \tag{1}$$

by right linear extension of forms,  $\phi_V(\omega \otimes v) = \omega(\beta(v))$ . It follows that

$$(\phi_V \otimes id) \circ (id \otimes \beta) = \Delta \circ \phi_V \tag{2}$$

hence  $\phi_V$  is a homomorphism of (right) comodules. From lemma 1,  $C$  is spanned by the coefficients of the full set of undecomposable injectives  $F$

$$C = \sum_{I \in F} \phi_I(I^* \otimes I). \tag{3}$$

Under favourable circumstances, e.g. if  $C$  is semisimple, there is a subfamily  $F' \subseteq F$  such that  $C = \bigoplus_{I \in F'} \phi_I(I^* \otimes I)$ . If furthermore  $\ker(\phi_I) = \{0\}$  for all  $I \in F'$  this decomposition of  $C$  is the Peter Weyl lemma.

Let  $\sigma(I) = I_0$ . Then there is a subfamily  $F' \subseteq F$  such that

$$\sigma(C) = \sum_{I \in F'} \phi_I(I_0^* \otimes I_0) \simeq \bigoplus_{I \in F'} m_I I_0. \quad (4)$$

Due to the grading, the multiplicities  $m_I$  do in general not equal  $\dim(I_0)$ . But, via  $\phi_I$ , every copy of  $I_0$  in  $C$  is contained in a copy of its injective cover. From (3) and lemma 1 it follows that

$$C \simeq \bigoplus_{I \in F'} m_I I \oplus X \quad (5)$$

where  $X$  is a submodule of  $C$ . Since  $\sigma(X) = \{0\}$  by (4),

$$C \simeq \bigoplus_{I \in F'} m_I I. \quad (6)$$

This is Green's structure theorem for coalgebras. In particular  $C$  is itself injective.

With the paradigm of compact Lie groups in mind this result can be seen as a decomposition of  $C$  in a direct sum of principal vector spaces of, loosely speaking, the centre of the dual algebra  $C^*$ . As a corollary, for instance, there is always a well-structured theory of Fourier analysis on compact Lie supergroups or quantizations thereof. The failure of the Peter Weyl lemma for these may be looked at as the non-diagonalizability of their algebras of Casimir operators, the undecomposable injectives playing the role of simultaneous principal vector spaces. These are undecomposable as comodules but clearly decomposable as principal vector spaces of a given Casimir operator.

Up to this point the implications are insensitive to any notion of a grading as long as the properties of the category of comodules that were essential in the proof of lemma 1 are not lost. Based on Schur's lemma and Burnside's theorem [8, 9] it is possible to be slightly more explicit about this decomposition in the present  $\mathbb{Z}_2$ -graded context when working over an algebraically closed field. In this case the commutant of a simple comodule is either one- or two-dimensional. Its even part is proportional to the identity and its odd part is spanned by a square root of minus the identity. The odd part is non-zero if and only if the comodule is reducible in the non-graded sense. Hence the space of coefficients of a reducible graded simple comodule faces additional constraints.

For a comodule  $W$  let  $W'$  be  $W$  with its grading reversed. Consider the simple  $C$ -comodules modulo their grading and fix a family  $\hat{F}$  of undecomposable injective covers with a distinguished grading such that each appears as a submodule of  $C$ . Let  $\hat{F}_1$  be the subfamily of injective covers of absolutely irreducibles in  $\hat{F} = \hat{F}_1 \dot{\cup} \hat{F}_2$ . An inspection of Burnside's theorem [8] yields

*Proposition 1.* A  $\mathbb{Z}_2$ -graded coalgebra  $C$  over an algebraically closed field decomposes as

$$C \simeq \bigoplus_{I \in \hat{F}_1} (\dim(\sigma(I)_0)I \oplus \dim(\sigma(I)_1)I') \oplus \bigoplus_{I \in \hat{F}_2} (\frac{1}{2} \dim(\sigma(I))I). \quad (7)$$

This decomposition has an interesting consequence. Since every comodule is contained in a direct sum of copies of  $C$ , undecomposable injectives come with precisely two distinct gradings, which are reverse to each other.

A key role in the investigation of the structure of a given coalgebra is played by induced comodules. Suppose  $D$  is another coalgebra with comultiplication  $\Delta'$  and counit  $\varepsilon'$  such

that  $\pi : C \rightarrow D$  is a surjective mapping of coalgebras, i.e.  $\pi$  is onto,  $\Delta' \circ \pi = (\pi \otimes \pi) \circ \Delta$  and  $\varepsilon' \circ \pi = \varepsilon$ . For a  $D$ -comodule  $\beta : V \rightarrow V \otimes D$  let

$$\text{ind}_D^C(V) = \{x \in V \otimes C \mid (\beta \otimes id)(x) = (id \otimes (\pi \otimes id)\Delta)(x)\}. \tag{8}$$

$\text{ind}_D^C(V)$  is a  $C$ -comodule with structure map  $\delta : \text{ind}_D^C(V) \rightarrow \text{ind}_D^C(V) \otimes C$ ,  $\delta = (id \otimes \Delta)|_{\text{ind}_D^C(V)}$ . It is straightforward to see that  $\delta$  is well defined. In the present context it is important to understand that  $C \simeq \text{ind}_D^C(D)$  as  $C$ -comodules. To find this consider the mapping  $(\pi \otimes id)\Delta : C \rightarrow D \otimes C$ . It is again straightforward to show that the image of  $(\pi \otimes id)\Delta$  sits within  $\text{ind}_D^C(D)$ . Due to  $\varepsilon' \circ \pi = \varepsilon$ ,  $\cdot(\varepsilon' \otimes id)$  is a left inverse to  $(\pi \otimes id)\Delta$ . Hence  $C \hookrightarrow \text{ind}_D^C(D)$ . To show that this mapping is onto as well let  $\{d_j\}_{j \in J} \subset D$  be a basis and  $\Delta'(d_k) = \sum_{j \in J} d_j \otimes D_k^j$  for all  $k \in J$ . Then  $x = \sum_{j \in J} d_j \otimes x^j \in \text{ind}_D^C(D)$  is equivalent to  $(\pi \otimes id)\Delta(x^k) = \sum_{j \in J} D_j^k \otimes x^j$  for all  $k \in J$ . Consequently

$$\begin{aligned} (\pi \otimes id)\Delta \cdot (\varepsilon' \otimes id)(x) &= (\pi \otimes id)\Delta \left( \sum_{j \in J} \varepsilon'(d_j)x^j \right) \\ &= \sum_{j \in J} \varepsilon'(d_j) \left( \sum_{k \in J} D_k^j \otimes x^k \right) \\ &= \sum_{k \in J} \left( \sum_{j \in J} \varepsilon'(d_j)D_k^j \right) \otimes x^k \\ &= x. \end{aligned}$$

Hence  $\cdot(\varepsilon' \otimes id)$  inverts  $(\pi \otimes id)\Delta$  and  $C \simeq \text{ind}_D^C(D)$ . If the condition  $\varepsilon' \circ \pi = \varepsilon$  were dropped,  $(\pi \otimes id)\Delta : C \rightarrow \text{ind}_D^C(D)$  would still be onto but in general not injective.

Now any decomposition of  $D$ , especially that into undecomposable injectives  $D \simeq \bigoplus_{\nu} m_{\nu} I_{\nu}$ , yields a decomposition of  $C$  as

$$C \simeq \bigoplus_{\nu} m_{\nu} \text{ind}_D^C(I_{\nu}). \tag{9}$$

Since direct summands of injectives are obviously injectives, this implies in particular that induction carries injectives to injectives. The study of induced modules forms the basic task in the investigation of the structure of the representation theory of coalgebras. This task may be a very complex one in general. Even in the case of Lie superalgebras and related coalgebras it is therefore advisable to consider examples that come with additional structures within their induced modules. This happens for type-I Lie superalgebras where the additional structures are filtrations by means of Kac modules.

### 3. The general linear supergroup

The general linear supergroup offers an interesting example for an illustration of the general theory. Induction from the base Lie group comes with special filtrations that allow for some insight into the substructure of the corresponding coalgebra. From now on all constructions are meant to live over the base field of complex numbers unless otherwise mentioned.

It is sensible to begin with polynomials. Let  $C(p, q)$  be the bialgebra that is generated by a supermatrix  $((c_j^i))$  of dimension  $(p + q) \times (p + q)$ . The multiplication is meant to be graded commutative and the comultiplication comes from ordinary matrix multiplication as  $\Delta(c_j^i) = \sum_k c_k^i \otimes c_j^k$ . The counit is defined from  $\varepsilon(c_j^i) = \delta_j^i$ .

At the same time  $((c_j^i))$  defines a comodule structure  $\beta : V \rightarrow V \otimes C(p, q)$  on a  $p + q$ -dimensional graded vector space with respect to a basis  $\{e_i\}$  by  $\beta(e_i) = \sum_j e_j \otimes c_i^j$ , the defining representation. Consequently the  $k$ -homogeneous part  $C(p, q)_k$  of  $C(p, q)$  is spanned by the coefficients of the comodule  $V^{\otimes k}$ . In particular  $C(p, q) \simeq \bigoplus_{k \geq 0} C(p, q)_k$  is a decomposition into finite-dimensional coalgebras. The transposition  $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$ , for homogeneous elements  $x \in V_{|x|}, y \in V_{|y|}$ , induces a representation of the permutation group  $\Sigma_k$  of  $k$  objects on  $V^{\otimes k}$ . Its enveloping algebra is the commutant of  $C(p, q)_k^*$ . Since the endomorphism ring of a semisimple module is semisimple this implies that  $C(p, q)_k$  is semisimple if the characteristic of the base field is either zero or exceeds  $\#(\Sigma_k) = k!$ . This is in complete analogy to the case of polynomial representations of  $GL_n$  [10]. Furthermore, in characteristic zero,  $C(p, q)$  is semisimple. This theory of polynomial representations of the general linear superalgebra is worked out in detail in [11]. It is worth to remark that the structure of this argument still holds for one parameter quantizations of  $C(p, q)$ . The major change is that the commutant of say the  $k$ -homogeneous part depends on the deformation parameter. If this dependence is reflected in defining relations that are polynomial in the parameter, inspection of its Killing form shows that the breakdown of semisimplicity is governed by zeros of polynomial equations in the deformation parameter. Hence one finds semisimplicity for polynomial representations of degree  $k$  in characteristic zero or greater than  $k!$  for all but finitely many values of the deformation parameter. Finally in characteristic zero a one parameter quantization of  $C(p, q)$  is generically semisimple apart from countably many values of the deformation parameter.

To pass to a supergroup,  $C(p, q)$  has to be enlarged to a Hopf algebra. To see how this can be done the inversion of supermatrices has to be considered. With respect to the natural block decomposition induced by the grading of  $V \simeq V_0 \oplus V_1$  the formal inverse of

$$((c_j^i)) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

reads

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}. \tag{10}$$

It follows that the matrix of generators  $((c_j^i))$  is invertible as soon as  $\det(A)$  and  $\det(D)$  are. To obtain a Hopf algebra  $A(p, q)$  with involutive antipode,  $C(p, q)$  has to be localized at the monoid which is generated by  $\det(A)$  and  $\det(D)$ . For notational convenience let

$$((c_j^i))^{-1} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}.$$

The structure of  $A(p, q)$  can be analysed with the help of induced modules with respect to the coalgebra  $A_0(p, q) \simeq A(p, 0) \otimes A(0, q)$ . The matrix of generators of  $A(p, 0)$  will be denoted by  $A_0$ , those of  $A(0, q)$  by  $D_0$ . The projection  $\pi : A(p, q) \rightarrow A_0(p, q)$  is defined componentwise by  $\pi(A) = A_0, \pi(D) = D_0, \pi(B) = 0, \pi(C) = 0$ . Let  $\beta : V \rightarrow V \otimes A_0(p, q)$  be a  $A_0(p, q)$ -comodule and  $\{e_i\} \subset V$  a basis such that  $\beta(e_i) = \sum_j e_j \otimes \mathcal{D}(A_0, D_0)_j^i$ . Now  $x = \sum_i e_i \otimes x^i \in \text{ind}_{A_0(p, q)}^{A(p, q)}(V)$  means  $(\pi \otimes id)\Delta(x^i) = \sum_j \mathcal{D}(A_0, D_0)_j^i \otimes x^j$ . This equation is straightforward to solve with the result  $x^i(A, B, C, D) = \sum_j \mathcal{D}(A, D)_j^i F^j(A^{-1}B, D^{-1}C)$ . Hence the elements of  $\text{ind}_{A_0(p, q)}^{A(p, q)}(V)$  are parametrized as

$$x(A, B, C, D) = \sum_{i,j} e_i \otimes \mathcal{D}(A, D)_j^i F^j(A^{-1}B, D^{-1}C) \tag{11}$$

and  $\dim(\text{ind}_{A_0(p,q)}^{A(p,q)}(V)) = 2^{2pq} \dim(V)$ . The explicit realization of the structure map

$$\delta_V : \text{ind}_{A_0(p,q)}^{A(p,q)}(V) \rightarrow \text{ind}_{A_0(p,q)}^{A(p,q)}(V) \otimes A(p, q) \tag{12}$$

reads

$$\delta_V(x) = \sum_{i,j,k} e_i \otimes \mathcal{D}(A_1, D_1)_j^i \mathcal{D}(A_2 + \xi_1 C_2, D_2 + \eta_1 B_2)_k^j \cdot \Delta(F^k(\xi, \eta)) \tag{13}$$

with  $\xi = A^{-1}B$ ,  $\eta = D^{-1}C$  and  $\Delta(\xi) = (A_2 + \xi_1 C_2)^{-1}(B_2 + \xi_1 D_2)$ ,  $\Delta(\eta) = (D_2 + \eta_1 B_2)^{-1}(C_2 + \eta_1 A_2)$ . For the convenience of notation the first and second factor in the tensor product after application of the comultiplication are distinguished by subscripts and identity matrices are not written explicitly.

There are two possibilities for similarity transformations, motivated by either distinguishing the variables  $\xi$  or  $\eta$  respectively. These arise from the identities

$$\Delta(A - BD^{-1}C) = \Delta(\tilde{A}^{-1}) = \tilde{A}_1^{-1}(\tilde{A}_2 - \tilde{B}_2 \eta_1)^{-1} \tag{14}$$

or

$$\Delta(D - CA^{-1}B) = \Delta(\tilde{D}^{-1}) = \tilde{D}_1^{-1}(\tilde{D}_2 - \tilde{C}_2 \xi_1)^{-1}. \tag{15}$$

Parametrizing

$$x(A, B, C, D) = \sum_{i,j} e_i \otimes \mathcal{D}(A, \tilde{D}^{-1})_j^i x_1^j(\xi, \eta) \tag{16}$$

the comodule structure is represented by

$$\delta_V(x(A, B, C, D)) = \sum_{i,j} e_i \otimes \mathcal{D}(A_1, \tilde{D}_1^{-1})_j^i \delta_V^1(x_1^j(\xi, \eta)) \tag{17}$$

with

$$\delta_V^1(x_1^j(\xi, \eta)) = \sum_k \mathcal{D}(A_2 + \xi_1 C_2, (\tilde{D}_2 - \tilde{C}_2 \xi_1)^{-1})_k^j \Delta(x_1^k(\xi, \eta)). \tag{18}$$

Correspondingly for

$$x(A, B, C, D) = \sum_{i,j} e_i \otimes \mathcal{D}(\tilde{A}^{-1}, D)_j^i x_2^j(\xi, \eta) \tag{19}$$

it follows that

$$\delta_V(x(A, B, C, D)) = \sum_{i,j} e_i \otimes \mathcal{D}(\tilde{A}_1^{-1}, D_1)_j^i \delta_V^2(x_2^j(\xi, \eta)) \tag{20}$$

with

$$\delta_V^2(x_2^j(\xi, \eta)) = \sum_k \mathcal{D}((\tilde{A}_2 - \tilde{B}_2 \eta_1)^{-1}, D_2 + \eta_1 B_2)_k^j \Delta(x_2^k(\xi, \eta)). \tag{21}$$

The substructure of these comodules is partly accessible by looking at suitable analogues of parabolic subgroups. The grading itself suggests a natural choice that was used by Kac in his original work on the finite representations of Lie superalgebras. To see how this looks in the present context one has to consider coalgebras generated by either the upper or lower block triangular entries of the matrix of generators. Hence there are coalgebras  $A_{+/-}(p, q)$  with projections  $\pi_{+/-} : A(p, q) \rightarrow A_{+/-}(p, q)$  given by  $\pi_{+/-}(A) = A_{+/-}$ ,  $\pi_{+/-}(D) = D_{+/-}$ ,  $\pi_+(B) = B_+$ ,  $\pi_-(B) = 0$ ,  $\pi_+(C) = 0$ ,  $\pi_-(C) = C_-$ . These allow for the notion of primitive elements. Let  $\gamma : W \rightarrow W \otimes A(p, q)$  be a comodule.  $x \in W$  is called primitive if  $(id \otimes \pi_+)\gamma(x) \in W \otimes A_+(p, q)$  is independent of  $B_+$ , antiprimitive if  $(id \otimes \pi_-)\gamma(x) \in W \otimes A_-(p, q)$  is independent of  $C_-$ .



In order to find the (anti)primitive elements in the induced modules consider

$$(id \otimes \pi_+) \delta_V^1(x_1^j(\xi, \eta)) = \sum_k \mathcal{D}(A_2, D_2)_k^i (id \otimes \pi_+) \Delta(x_1^k(\xi, \eta)) \tag{22}$$

and

$$(id \otimes \pi_-) \delta_V^2(x_2^j(\xi, \eta)) = \sum_k \mathcal{D}(A_2, D_2)_k^j (id \otimes \pi_-) \Delta(x_2^k(\xi, \eta)). \tag{23}$$

It remains for us to solve the equations

$$\begin{aligned} (id \otimes \pi_+) \Delta(f_+(\xi, \eta)) &= f_+(A_{+2}^{-1}(B_{+2} + \xi_1 D_{+2}), (D_{+2} + \eta_1 B_{+2})^{-1} \eta_1 A_{+2}) \\ &= f_+(A_{+2}^{-1} \xi_1 D_{+2}, D_{+2}^{-1} \eta_1 A_{+2}) \end{aligned} \tag{24}$$

and

$$\begin{aligned} (id \otimes \pi_-) \Delta(f_-(\xi, \eta)) &= f_-((A_{-2} + \xi_1 C_{-2})^{-1} \xi_1 D_{-2}, D_{-2}^{-1}(C_{-2} + \eta_1 A_{-2})) \\ &= f_-(A_{-2}^{-1} \xi_1 D_{-2}, D_{-2}^{-1} \eta_1 A_{-2}). \end{aligned} \tag{25}$$

The result is immediate from the formal substitution of  $B_{+2} = -\xi_1 D_{+2}$  or  $C_{-2} = -\eta_1 A_{-2}$ , since the corresponding equations do not depend on these variables

$$f_+(\xi, \eta) = f_+(0, (1 - \eta\xi)^{-1} \eta) \quad f_-(\xi, \eta) = f_-((1 - \xi\eta)^{-1} \xi, 0). \tag{26}$$

Finally a general primitive element of  $\text{ind}_{A_0(p,q)}^{A(p,q)}(V)$  has the form

$$x_+(A, B, C, D) = \sum_{i,j} e_i \otimes \mathcal{D}(A, \tilde{D}^{-1})_j^i F_+^j((1 - \eta\xi)^{-1} \eta) \tag{27}$$

and a general antiprimitive element correspondingly

$$x_-(A, B, C, D) = \sum_{i,j} e_i \otimes \mathcal{D}(\tilde{A}^{-1}, D)_j^i F_-^j((1 - \xi\eta)^{-1} \xi). \tag{28}$$

A short calculation establishes the relations

$$\Delta((1 - \xi\eta)^{-1} \xi) = \tilde{A}_2(1 + B_2 D_2^{-1} \eta_1) \{B_2 D_2^{-1} + (1 - \xi_1 \eta_1)^{-1} \xi_1 (1 + \eta_1 B_2 D_2^{-1})\} D_2 \tag{29}$$

$$\Delta((1 - \eta\xi)^{-1} \eta) = \tilde{D}_2(1 + C_2 A_2^{-1} \xi_1) \{C_2 A_2^{-1} + (1 - \eta_1 \xi_1)^{-1} \eta_1 (1 + \xi_1 C_2 A_2^{-1})\} A_2. \tag{30}$$

Hence, using the parameterization

$$x(A, B, C, D) = \sum_{i,j} e_i \otimes \mathcal{D}(A, \tilde{D}^{-1})_j^i x_1^j(\xi, (1 - \eta\xi)^{-1} \eta) \tag{31}$$

it follows from equation (30) that a filtration of  $\text{ind}_{A_0(p,q)}^{A(p,q)}(V)$  is inherited from the expansion with respect to the composite variables  $(1 - \eta\xi)^{-1} \eta$ . That means for every  $j \leqq pq$  let  $W_j(V)$  be the subcomodule of  $\text{ind}_{A_0(p,q)}^{A(p,q)}(V)$  spanned by all elements with  $x_1^k$  of degree less or equal than  $j$  with respect to the entries of  $(1 - \eta\xi)^{-1} \eta$ . Consequently

$$\{0\} \subset W_0(V) \subset W_1(V) \subset \dots \subset W_{pq}(V) = \text{ind}_{A_0(p,q)}^{A(p,q)}(V) \tag{32}$$

and from the above relations  $W_j(V)/W_{j-1}(V) \simeq W_0(V \otimes V_j)$  with  $V_j$  the  $A_0(p, q)$ -comodule spanned by the  $j$ th order monomials of  $(1 - \eta\xi)^{-1} \eta$ . The invariant subspace  $W_0(V)$ , corresponding to degree zero in  $(1 - \eta\xi)^{-1} \eta$ , is spanned by elements of the form

$$x(A, B, C, D) = \sum_{i,j} e_i \otimes \mathcal{D}(A, \tilde{D}^{-1})_j^i x_1^j(\xi). \tag{33}$$

The only primitive vectors contained in  $W_0(V)$  are those with  $x_1^k(\xi)$  independent of  $\xi$ . Obvious antiprimitive vectors in these representations are those with all  $x_1^k$  proportional to the highest power in the entries of  $\xi$ ; however it may happen that there are more of them, implying that  $W_0(V)$  may be reducible but indecomposable. These comodules correspond to one type of Berezin's elementary representations [8] which are in turn equivalent to lowest weight Kac modules [12, 13]. The irreducible ones correspond to typical irreducible modules in Kac's terminology.

Analogously

$$x(A, B, C, D) = \sum_{i,j} e_i \otimes \mathcal{D}(\tilde{A}^{-1}, D)_j^i x_2^j((1 - \xi\eta)^{-1}\xi, \eta) \tag{34}$$

leads to a filtration of  $\text{ind}_{A_0(p,q)}^{A(p,q)}(V)$  coming from the expansion with respect to the variables  $(1 - \xi\eta)^{-1}\xi$ . The invariant subspace corresponding to degree zero is in this case spanned by elements of the form

$$x(A, B, C, D) = \sum_{i,j} e_i \otimes \mathcal{D}(\tilde{A}^{-1}, D)_j^i x_2^k(\eta). \tag{35}$$

These comodules correspond to the second type of Berezin's elementary representations which are basically highest weight Kac modules. The only antiprimitive elements contained are those with  $x_2^k(\eta)$  independent of  $\eta$ . There are primitive elements with all  $x_2^k$  proportional to the highest power in the entries of  $\eta$  and the comodules correspond to highest weight Kac modules of the general linear Lie superalgebra. Again these representations are typical in the sense of Kac if irreducible.

To conclude this section, let  $V \subset A_0(p, q)$  be a subcomodule with its basis elements  $e_i = E_i(A_0, D_0)$ . Then

$$\sum_i \varepsilon'(e_i) \mathcal{D}(X, Y)_j^i = \sum_i E_i(1, 1) \mathcal{D}(X, Y)_j^i = E_j(X, Y) \tag{36}$$

and as elements of  $A(p, q)$  the above choices of coordinates read either

$$x(A, B, C, D) = \sum_i E_i(A, D(1 - \eta\xi)) x_1^i(\xi, (1 - \eta\xi)^{-1}\eta) \tag{37}$$

or

$$x(A, B, C, D) = \sum_i E_i(A(1 - \xi\eta), D) x_2^i((1 - \xi\eta)^{-1}\xi, \eta) \tag{38}$$

since  $\tilde{A}^{-1} = A(1 - \xi\eta)$  and  $\tilde{D}^{-1} = D(1 - \eta\xi)$ . Here the notation  $x$  is abused to represent  $(\varepsilon' \otimes id)x$ . The explicit splitting into indecomposable injectives has to be derived from the information about the (anti)primitive elements and the associated filtrations, which is still a formidable task when considered in full generality. Introducing weight spaces and using Kac's results about Casimir elements it is easy to verify, that the elementary invariant subspaces  $W_0(V)$  split from  $\text{ind}_{A_0(p,q)}^{A(p,q)}(V)$  iff they are typical irreducible. However, in the atypical cases the corresponding indecomposable injectives are strictly larger than these submodules as is to be illustrated below.

#### 4. Examples

Finally the general theory is to be illustrated by simple examples. Fortunately the easiest case, i.e  $A(1, 1)$ , which describes a part of the locally finite modules of the Lie superalgebra  $gl(1, 1; \mathbb{C})$ , shows already the general features without demanding involved computations. Therefore it is well suited for illustrative purposes. This should at the same time clarify

Therefore it is well suited for illustrative purposes. This should at the same time clarify confusions that arose in the literature concerning Fourier analysis on the unitary supergroup  $U(1, 1)$  [14].

To begin with, the underlying bosonic coalgebra  $A_0(1, 1)$  is generated by two independent bosonic variables  $A_0$  and  $D_0$  and their inverses. The decomposition of  $A_0(1, 1)$  corresponds to its  $\mathbb{Z} \times \mathbb{Z}$  grading. The simple comodules  $V_{n_1, n_2}$  are one-dimensional and labelled by two integers  $n_1, n_2$ . As a basis of  $V_{n_1, n_2}$  within  $A_0(1, 1)$  choose  $A_0^{n_1} D_0^{n_2} \in A_0(1, 1)$ . With respect to the parameterization (37), elements of  $\text{ind}_{A_0(1,1)}^{A(1,1)}(V_{n_1, n_2}) \subset A(p, q)$  are written as

$$x(A, B, C, D) = \tilde{A}^{-n_1} D^{n_2} x_2((1 - \xi\eta)^{-1}\xi, \eta). \tag{38}$$

The primitive vectors in this representation are given by

$$x_+(A, B, C, D) = A^{n_1} \tilde{D}^{-n_2} F_+((1 - \eta\xi)^{-1}\eta) \tag{39}$$

$$= \tilde{A}^{-n_1} D^{n_2} (1 - \eta\xi)^{n_1+n_2} F_+(\eta). \tag{40}$$

Obviously the cases  $n_1 + n_2 = 0$  bear a significant difference compared to  $n_1 + n_2 \neq 0$ ; all primitive elements live inside the elementary subcomodule which as a consequence cannot split from the induced representation. Hence  $n_1 + n_2 = 0$  marks four-dimensional injective modules that extend one-dimensional irreducible representations, which are atypical in Kac's terminology. Their Jordan Hölder sequences have length four. In the case of  $n_1 + n_2 \neq 0$  the comodule  $\text{ind}_{A_0(1,1)}^{A(1,1)}(V_{n_1, n_2})$  splits into the direct sum of two typical irreducible representations spanned by either the elements

$$y_1^{n_1, n_2}(A, B, C, D) = \tilde{A}^{-n_1} D^{n_2} \cdot \eta \tag{41}$$

$$y_2^{n_1, n_2}(A, B, C, D) = \tilde{A}^{-n_1} D^{n_2} \cdot 1 \tag{42}$$

or

$$y_3^{n_1, n_2}(A, B, C, D) = \tilde{A}^{-n_1} D^{n_2} \cdot (1 - \eta\xi)^{n_1+n_2} \tag{43}$$

$$= \tilde{A}^{-n_1} D^{n_2} \cdot (1 - (n_1 + n_2)\eta\xi)$$

$$y_4^{n_1, n_2}(A, B, C, D) = \tilde{A}^{-n_1} D^{n_2} \cdot \xi \tag{44}$$

respectively. In the case of  $n_1 + n_2 = 0$  the basis vector  $y_3$  may be replaced by

$$y_3^{n_1, n_2'}(A, B, C, D) = \tilde{A}^{-n_1} D^{n_2} \cdot \eta\xi. \tag{45}$$

This completes the explicit construction of the Fourier analysis of  $A(1, 1)$  and, by specialization, that of the compact unitary supergroup  $U(1, 1)$ . It is easy to verify explicitly, that the standard quadratic Casimir element of  $U(\mathfrak{gl}(1, 1; \mathbb{C}))$  is proportional to the identity on the two-dimensional injectives corresponding to  $n_1 + n_2 \neq 0$  and falls into two one-dimensional and one two-dimensional Jordan blocks when acting on one of the four-dimensional injectives with  $n_1 + n_2 = 0$ . The latter are all degenerate and form principal vector spaces that extend the kernel of the Casimir operator.

The next step should be an investigation of  $A(2, 1)$ . Since a lot has been published about the Lie superalgebras  $\mathfrak{gl}(2, 1; \mathbb{C})$  and  $\mathfrak{sl}(2, 1; \mathbb{C})$ , respectively, this case will be treated by an informal discussion of its features, relating it to the literature. The structure of the induced modules is slightly more involved than in the  $A(1, 1)$  example. In terms of elementary representations the Kac filtrations of the induced comodules have four composition factors. The introduction of weights and inspection of those of the primitive elements shows that reducible elementary representations always occur in pairs as composition factors of induced

representations. This leads to a picture that is very similar to the results concerning  $A(1, 1)$ . Indecomposable injectives are either typical irreducible representations or they extend atypical irreducible representations and then have Kac filtrations of length two. These results are contained in reference [15], where they were derived in a Lie superalgebra context, without realising that the matter of investigation was the coalgebra encoding the locally finite representations.

## 5. Conclusion

The representation theory of graded coalgebras has been discussed, in particular shedding some light on the role of atypical irreducible representations of type-I Lie superalgebras. The basic problem for further advances is the classification of their injectives. It would be interesting if that could be carried out at least for all  $A(p, q)$ . Partial results come from the construction of the (anti)primitive elements in induced representations. Others can be deduced from Kac's original work on the characters of simple Lie superalgebras, especially to obtain information about the blocks [5] of, e.g.,  $A(p, q)$ . One might speculate as to whether the injective comodules of  $A(p, q)$  could be characterized by the property to have Kac filtrations by means of both types of elementary representations, as is in the case with  $A(1, 1)$  and  $A(2, 1)$ . If this is not the case, a counterexample would be of interest. I hope to come back to these questions once the details piece together to provide satisfying additional insight.

## Acknowledgments

I would like to thank Professor M R Zirnbauer for discussions as well as Dr N Andruskiewitsch, who furthermore brought reference [15] to my attention. Professor S Donkin pointed references [5, 10, 11] out to me. This work had its origin within the SFB 341 Köln–Aachen–Jülich and was completed with the financial support of the EC.

## References

- [1] Zirnbauer M R 1994 unpublished
- [2] Zirnbauer M R 1991 Fourier analysis on a hyperbolic supermanifold with constant curvature *Commun. Math. Phys.* **141** 503–22
- [3] Zirnbauer M R 1990 Fourier inversion theorem in mesoscopic transport *Physica* **167A** 132–9
- [4] Zirnbauer M R 1992 Super Fourier analysis and localization in disordered wires *Phys. Rev. Lett.* **69** 1584–7
- [5] Green J A 1976 Locally finite representations *J. Algebra* **41** 137–71
- [6] Waterhouse W C 1979 *Introduction to Affine Group Schemes (Graduate Texts in Mathematics 66)* (Berlin: Springer)
- [7] Manin Y 1991 *Topics in Noncommutative Geometry* (Princeton: Princeton University Press)
- [8] Berezin F A 1987 *Introduction to Superanalysis* (Dordrecht: Reidel)
- [9] Scheunert M 1985 Selected topics from the representation theory of Lie superalgebras *Supersymmetry (Bonn, 1984)* (New York: Plenum)
- [10] Green J A 1981 *Polynomial Representations of  $GL(n)$  (Lecture Notes in Mathematics 830)* (Berlin: Springer)
- [11] Muir N J 1991 Polynomial representations of the general linear Lie superalgebra *PhD Thesis* Queen Mary & Westfield College, London
- [12] Kac V G 1977 Characters of typical representations of classical Lie superalgebras *Commun. Algebra* **5** 889
- [13] Kac V G 1978 *Representations of Classical Lie Superalgebras (Lecture Notes in Mathematics 676)* (Berlin: Springer) pp 597–626
- [14] Guhr T 1993 On the graded group  $U(1/1)$  *J. Math. Phys.* **34** 2541–53
- [15] Su Y 1992 Classification of finite-dimensional modules of the Lie superalgebra  $sl(2, 1)$  *Commun. Algebra* **20** 3259–77